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DESIGN OF STATICALLY DETERMINATE TRUSSES FOR MINIMUM WEIGHT AND DEFLECTION

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ABSTRACT

The stiffness/weight ratios of statically determinant plane or space trusses are maximized by adjusting their bar areas and by optimizing their configurations. When minimum bar areas are specified together with the outline of a truss, a simple nonlinear programming problem is obtained which yields a global optimum. In the pure deflection problem where minimum bar areas are not assigned, three cases are encountered. In one, no physical solutions exist; in the second, a unique set of bar areas are obtained which represent the absolute minimum weight design for a specified deflection or conversely; and in the last, a degenerate case is obtained in which positive, negative, or zero deflection can be achieved at a node with an infinite number of truss designs of vanishing weight. Under very special circumstances the minimum deflection trusses display uniform stresses. Here, the optimum truss configuration corresponds to a Michell structure designed for equal tensile and compressive stresses. In general, however, the truss outline may be adjusted to produce the degenerate case in which any deflection is obtainable with structures of arbitrarily small weight.

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DESIGN OF STATICALLY DETERMINATE TRUSSES FOR MINIMUM WEIGHT AND DEFLECTION

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I. INTRODUCTION

As materials of ever increasing strength are made available, the proportioning of structural components will be governed more and more by stiffness and stability rather than strength. This report addresses itself to the problem of designing a statically determinate plane or space truss under a single load system so that its stiffness/weight ratio is maximum. This may be accomplished by optimizing the location of the truss nodes and by optimizing the distribution of bar areas; both procedures are treated.

Specifying the truss outline together with certain member sizes, our first studies required that the "open" bar areas in the truss be varied to produce a given node deflection with a minimum volume of material. Depending on the loading and the specified deflection, three situations were encountered. In one, no physical bar areas exist which will satisfy the deflection requirement. In another, one finds an infinite number of bar area distributions which produce not only the specified deflection, but also, negative deflections or zero deflection at the given node. For this case, when strength and stability are disregarded, any deflection can be achieved with trusses of vanishing weight. In the final case, a unique bar area distribution is obtained which represents the absolute minimum weight design of the truss. One of the characteristics of this truss is that the product of the actual stress and the virtual stress is the same for all bars; the virtual stresses arise from a unit load placed in the direction and at the node of the desired deflection. Designs of minimum weight and uniform stress beams and trusses are compared for equal constant depth members. The beam is found to be superior to the truss on a strictly stiffness/weight basis.

In our second study, we again consider a truss of fixed outline with certain member sizes specified; only here, the open bar areas are required to be no smaller than certain specified minimum areas established from perhaps code, strength, or stability requirements. If the deflections of such a truss are excessive when the bar areas are taken as their minimum specified values, a method is presented for stiffening the truss with a minimum increase in the weight. The associated mathematical problem may be formulated as a nonlinear programming problem with a nonlinear objective function and linear constraints. A very rapid procedure suitable for a desk calculator is described for finding the exact solution to this problem in a finite number of iterations. The resulting truss is unique and represents the absolute minimum weight design producing a specified node deflection.

In our final minimum weight design problem, both the location of the truss nodes and the bar areas are allowed to vary. For this problem, previous investigators have concluded that the optimum stiffness/weight truss is a Michell structure. We show that this uniformly stressed structure is optimum only when the actual and virtual loadings are proportional. When this is not the case, an infinite number of truss configurations can be found which produce any desired node deflection, including zero deflection, with a structure of vanishing weight.

II. TRUSSES WITH GIVEN CONFIGURATIONS

The deflection of any joint of a pin-connected truss is given by the virtual work expression

$$\Delta = \sum \frac{SuL}{A\,E} \tag{1}$$

where, for any member, S is the direct stress resulting from the applied loading, u is the direct stress resulting from a unit load applied in the specified direction at the joint where the defelction is desired, L is the length, A is the area, and E is the modulus of elasticity. The summation extends over all bars in the structure.

Consider the bars in a statically determinate truss to be divided into two groups. In the first, the members will be completely described and denoted by the subscript c(closed). In the second group, everything except the member areas will be specified and these will be treated as open parameters. This group will be denoted by the subscript o. Equation (1) may be rewritten as

$$\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c} = \sum_o \frac{S_o u_o L_o}{A_o E_o} \tag{2}$$

where the symbols $\sum_c$ and $\sum_o$ mean the summation over the members of group c and summation over group o respectively. Since we are considering a design problem as opposed to an analysis problem, the deflection Δ will be specified and the areas A_o will be sought. It is then meaningful to distinguish four cases. These will be treated in the following sections.

Case 1: The sign of the product $S_o u_o$ is either non-positive or non-negative for all members and the left side of Eq. (2) is zero; or analytically,

$$\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c} = 0$$ and either $S_o u_o \geq 0$ or $S_o u_o \leq 0$ for all members o.

For this case, Eq. (2) cannot be satisfied using only finite values for A_o. Thus, no physical solution exists.

Case 2: The sign of the product $S_o u_o$ is different from that of the left side of Eq. (2) for all members; or analytically,

$$\frac{S_o u_o}{\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c}} \leq 0 \quad \text{for all members o.}$$

For this condition, the signs of the right and left side of Eq. (2) cannot be made the same unless negative values of the areas A_o are admitted. Again, no physical solution exists.

Case 3: The product $S_o u_o$ is positive for some truss members and negative for others,

$$S_o u_o < 0 \quad \text{and} \quad S_o u_o > 0 \quad \text{for the members o.}$$

We shall show that in this case any specified deflection value can be obtained using a truss of arbitrarily small weight. Let each open member with a negative product $S_o u_o$ have an area A_1, and let the members with a positive product $S_o u_o$ have an area A_2. Then, Eq. (2) may be written as

$$\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c} = \frac{1}{A_2} \sum_{+o} \frac{S_o u_o L_o}{E_o} - \frac{1}{A_1} \sum_{-o} \left| \frac{S_o u_o L_o}{E_o} \right| \qquad (3)$$

where the symbols $\sum_{+o}$ and $\sum_{-o}$ are the sums over the members with positive and negative products $S_o u_o$ respectively. Solving Eq. (3) for A_1 we find that any finite deflection Δ can be achieved with non-negative areas A_1 and A_2 when A_1 is given by

$$A_1 = \frac{\sum_o \frac{S_o u_o L_o}{E_o}}{\frac{1}{A_2} \sum_{+o} \frac{S_o u_o L_o}{E_o} - \left(\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c}\right)} \tag{4}$$

and A_2 is taken sufficiently small. It is clear that

$$\lim_{A_2 \to 0} A_1 = 0 \tag{5}$$

Physically, we note that the stresses and deflections at joints other than the one with the specified deflection approach infinity as the two areas A_1 and A_2 approach zero.

It can be seen from Eq. (3) that by "beefing up" members with negative products $S_o u_o$ (increasing A_1), the deflection is increased. On the other hand, by making such members more flexible we produce unusual effects such as upward deflections of simply supported trusses under downward acting loads. This situation is illustrated in the photograph shown in Fig. 1. If the flexibility of members with $S_o u_o < 0$ are adjusted so that zero deflection is obtained at the specified node, this condition will persist as the loading is increased or decreased proportionally.

Case 4: The sign of the product $S_o u_o$ is the same as that of the left side of Eq. (2) for all members; or analytically,

$$\frac{S_o u_o}{\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c}} \geq 0 \text{ for all members } o.$$

The numerator and denominator in the above fraction will be considered, without loss in generality, as non-negative quantities. For this case, we shall find a set of bar areas A_o^* which minimize the truss weight subject to the condition that Δ is a specified constant.

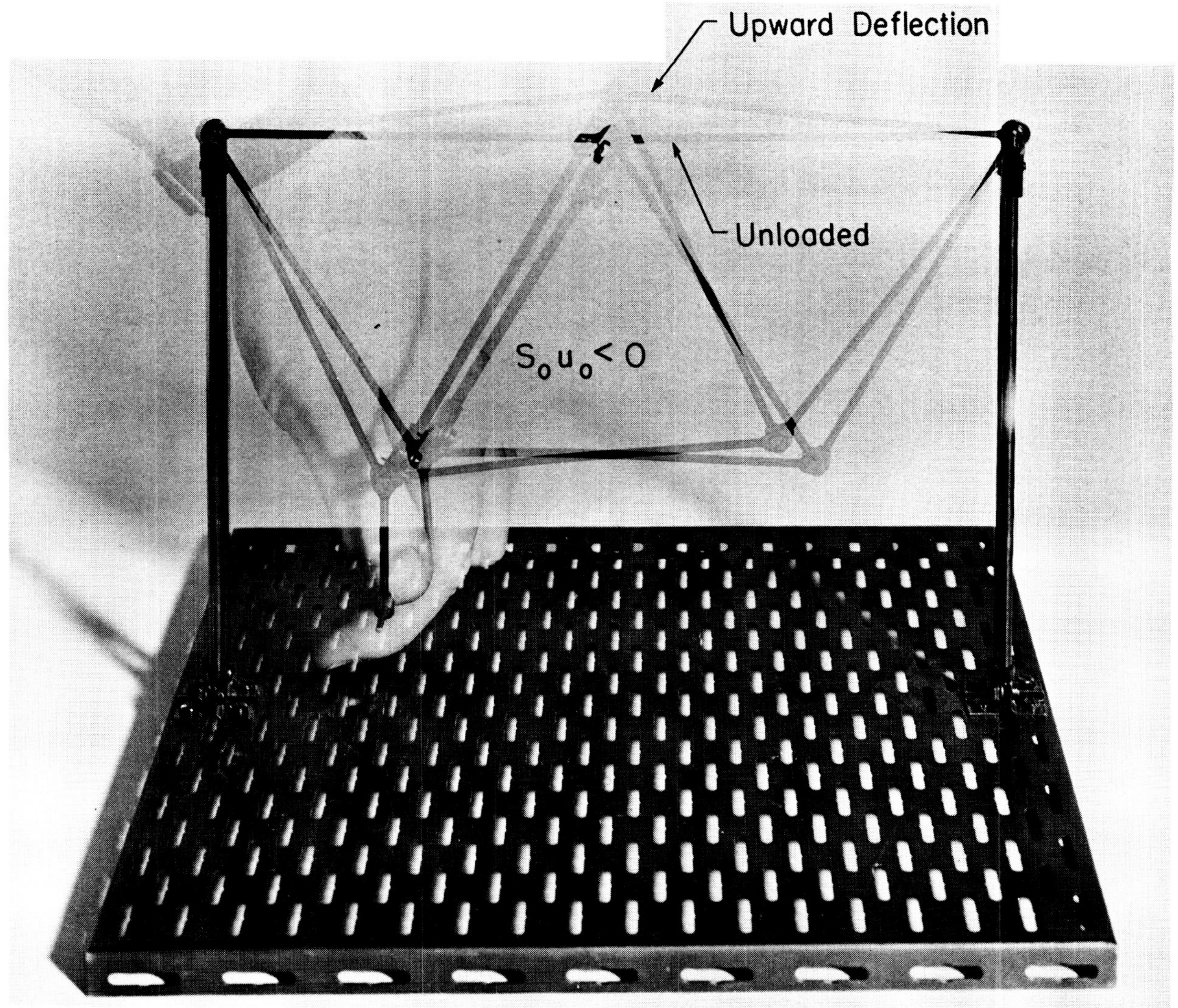


Fig. 1 TRUSS EXHIBITING UPWARD DEFLECTION DUE TO A DOWNWARD ACTING LOAD

The weight of a truss may be written

$$W = \sum_c \rho_c A_c L_c + \sum_o \rho_o A_o L_o \tag{6}$$

where ρ is the weight density of a bar. Repeating Eq. (2), the deflection is

$$\Delta = \sum_c \frac{S_c u_c L_c}{A_c E_c} + \sum_o \frac{S_o u_o L_o}{A_o E_o} = \text{specified constant.} \tag{7}$$

Using Lagrange's method of undetermined multipliers, a set of areas A_o^* may be found from Eqs. (6) and (7) which render W stationary; hence,

$$\frac{\partial}{\partial A_o}\left[\sum_c \rho_c A_c L_c + \sum_o \rho_o A_o L_o + \lambda\left(\sum_c \frac{S_c u_c L_c}{A_c E_c} + \sum_o \frac{S_o u_o L_o}{A_o E_o}\right)\right] = 0 \tag{8}$$

where λ is the Lagrangian multiplier. Performing the operations in Eq. (8), solving for A_o, and eliminating λ by Eq. (7), A_o^* becomes

$$A_o^* = \frac{\sqrt{\frac{S_o u_o}{\rho_o E_o}} \sum_o \left(\frac{S_o u_o \rho_o}{E_o}\right)^{1/2} L_o}{\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c}} \tag{9}$$

The weight W^* associated with A_o^* may be found by substituting Eq. (9) into (6):

$$W^* = \sum_c \rho_c A_c L_c + \frac{\left[\sum_o \left(\frac{S_o u_o}{E_o/\rho_o}\right)^{1/2} L_o\right]^2}{\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c}} \tag{10}$$

We shall now show that the stationary value W^* is an absolute minimum; i.e., for any set of areas $A_o \geq 0$ satisfying Δ = specified constant, $W^* \leq W$.

Define $F \equiv W - W^*$

Using Eqs. (6) and (10), F becomes

$$F = \sum_c \rho_c A_c L_c + \sum_o \rho_o A_o L_o - \left\{ \sum_c \rho_c A_c L_c + \frac{\left[\sum_o \left(\frac{S_o u_o \rho_o}{E_o} \right)^{1/2} L_o \right]^2}{\Delta - \sum_c \frac{S_c u_c L_c}{A_c E_c}} \right\} \tag{11}$$

Substituting for Δ from Eq. (7), one obtains

$$F = \frac{1}{\sum_o \frac{S_o u_o L_o}{A_o E_o}} \left\{ \left(\sum_o \frac{S_o u_o L_o}{A_o E_o} \right) \left(\sum_o \rho_o A_o L_o \right) - \left[\sum_o \left(\frac{S_o u_o L_o}{A_o E_o} \right)^{1/2} \left(\rho_o A_o L_o \right)^{1/2} \right]^2 \right\} \tag{12}$$

The quantity in braces is non-negative by Schwarz's inequality. Since the quantity

$$\sum_o \frac{S_o u_o L_o}{A_o E_o}$$

is also non-negative, $F \geq 0$. Q.E.D.

It should be noted that the weight of the "open" members described by the second term in Eq. (10) is inversely proportional to the specific stiffness E/ρ. This ratio is approximately equal to 10^8 in. for most of the common metals; for ceramics we find specific stiffnesses as high as 10^9 in.

If a truss is designed using one material, Eq. (9) indicates that in the optimum stiffness/weight truss the product of the actual stress and the virtual stress is constant over all the open members, i.e.,

$$\left(\frac{S_o}{A}\right)\left(\frac{u_o}{A}\right) = \left[\frac{E\Delta - \sum_c \frac{S_c u_c L_c}{A_c}}{\sum_o \left(S_o u_o\right)^{1/2} L_o}\right]^2 = \text{constant} \tag{13}$$

When the actual and virtual loadings are proportional, $S=ku$ where k is a constant. For such cases it is evident from Eq.(13) that the optimum deflection design is a uniformly stressed truss.

In Fig. 2 weight comparisons are made among minimum weight beams and trusses and uniform stress trusses when the designs are based on deflection. The detailed weight relationships for these members are developed in Appendix C. The design of optimum stiffness/weight beams is outlined in Appendix A; the weight of a uniform stress truss designed on the basis of deflection is given in Appendix B.

For low values of L/d where shear deformations are significant, a beam is found to be far superior to a truss when the designs are based on stiffness. For large values of L/d, most of the truss weight is concentrated in the chords to resist bending deformation. The resulting trusses are quite similar to webless I-beams and the comparisons drawn for this "ideal" member in Table 1 and Fig. 4 of Ref. 1 hold exactly for the trusses when their L/d approach infinity.

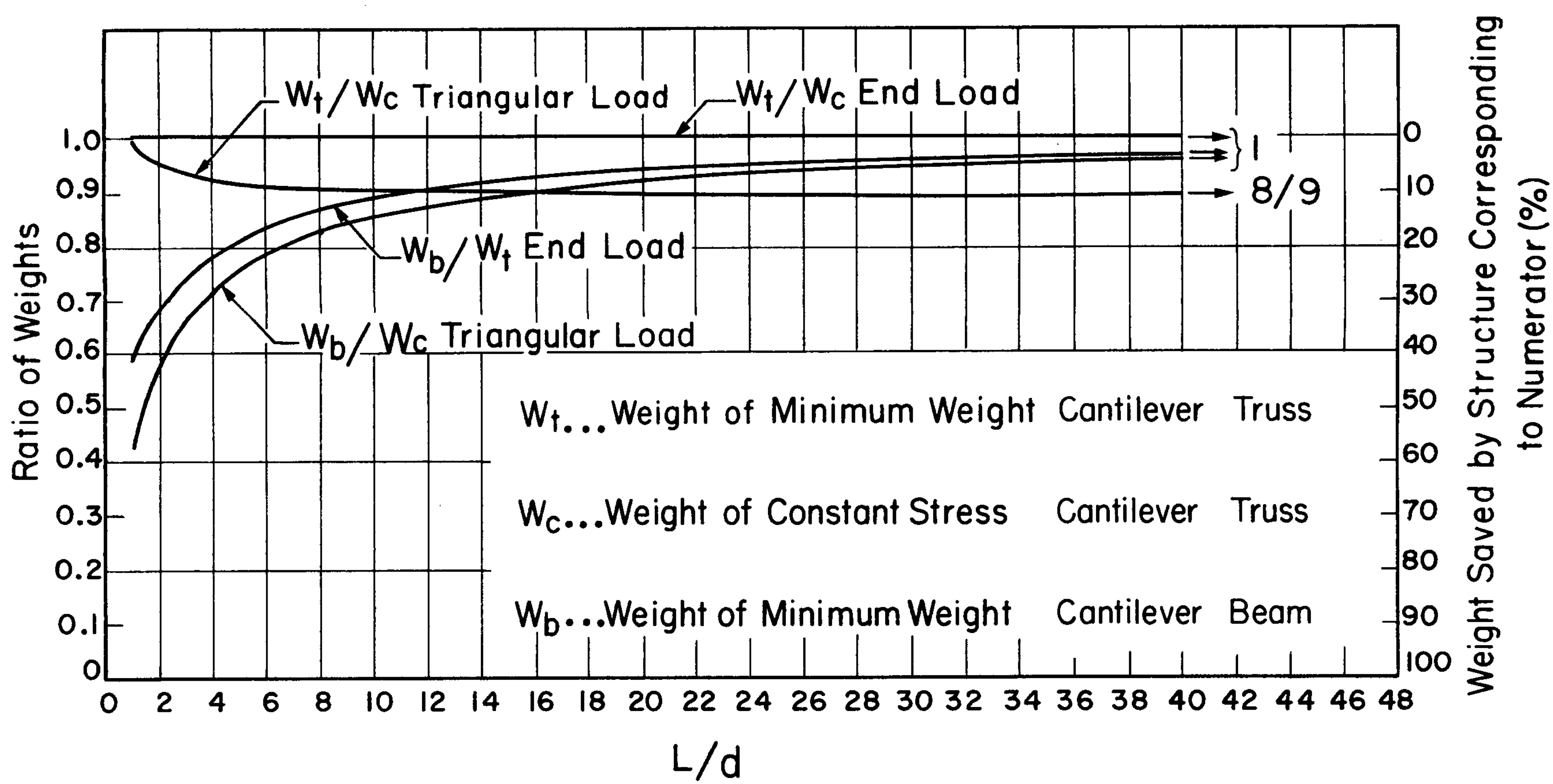


Fig. 2 COMPARISONS AMONG MINIMUM WEIGHT BEAMS AND TRUSSES AND UNIFORM STRESS TRUSSES

III. TRUSSES WITH GIVEN CONFIGURATIONS AND MINIMUM BAR SIZES

A. Design Algorithm

Of the four cases discussed in Section II, only the third and fourth gave rise to physically attainable solutions. In both of these cases the optimum areas for the open members were established without considering the consequences of strength, stability, corrosion, or code requirements. In this section we shall again examine cases three and four; but this time, the open members will be assigned minimum areas based on such criteria.

Consider a truss loaded in such a manner that some of the open members have positive products $S_o u_o$ and some negative. Suppose that the deflection at some joint of this truss is excessive when minimum bar areas A_m are used in the members. Since the specified deflection Δ can always be taken as non-negative, the foregoing situation can be expressed as

$$0 \leq \Delta < \sum_{+o} \frac{S_o u_o L_o}{E_o A_m} - \sum_{-o} \left| \frac{S_o u_o L_o}{E_o A_m} \right| + \sum_{c} \frac{S_c u_c L_c}{E_c A_c} \qquad (14)$$

With only the areas of the open members at our disposal, we must try to reduce the magnitude of the right side of this inequality to the specified value Δ. Clearly, the magnitude is reduced when the areas of the (+ O) group are increased and when the areas of the (- O) group are decreased. The latter course is preferred since it is accompanied by a decrease in the truss weight. However, in Eq. (14) the members in the (- O) group have already the smallest admissible areas. The remaining possibility is to increase the areas of the (+ O) group with a corresponding increase in truss weight.

There are circumstances in which the flexibility of the closed members are so great that the specified deflection cannot physically be achieved. The condition for the existence of a solution is found from Eq. (14) when the areas of the (+ 0) group are allowed to approach infinity; hence,

Case 5:

$$\Delta - \sum_{c} \frac{S_c u_c L_c}{E_c A_c} + \sum_{-o} \left| \frac{S_o u_o L_o}{E_o A_m} \right| \leq 0 \qquad (15)$$

physical solution is impossible.

Case 6:

$$\Delta - \sum_{c} \frac{S_c u_c L_c}{E_c A_c} + \sum_{-o} \left| \frac{S_o u_o L_o}{E_o A_m} \right| > 0 \qquad (16)$$

physical solutions are possible.

Infinitely many solutions exist when the inequality of Eq. (16) holds; however, only one minimizes the truss weight. The following procedure stiffens a truss with a minimum increase in weight.

(1) Let the minimum areas be used for those members in which the product $S_o u_o$ is negative. These members should now be included in group c.

(2) Treat the remaining areas as open parameters (group o) and determine their magnitudes from Eq. (9).

(3) If any of the areas A_o^* assume values lower than their minimum values, increase their magnitudes to their minimum values and transfer them to group c.

(4) Return to Step 2 and repeat the process until the areas determined by Eq. (9) are all greater than or equal to their minimum values.

After demonstrating this design procedure by the following example, we shall return to Step 3 and comment on its validity.

B. Example

Design the truss shown in Fig. (3-a) so that the downward deflection at joint G is equal to 55a/E and the weight is a minimum.

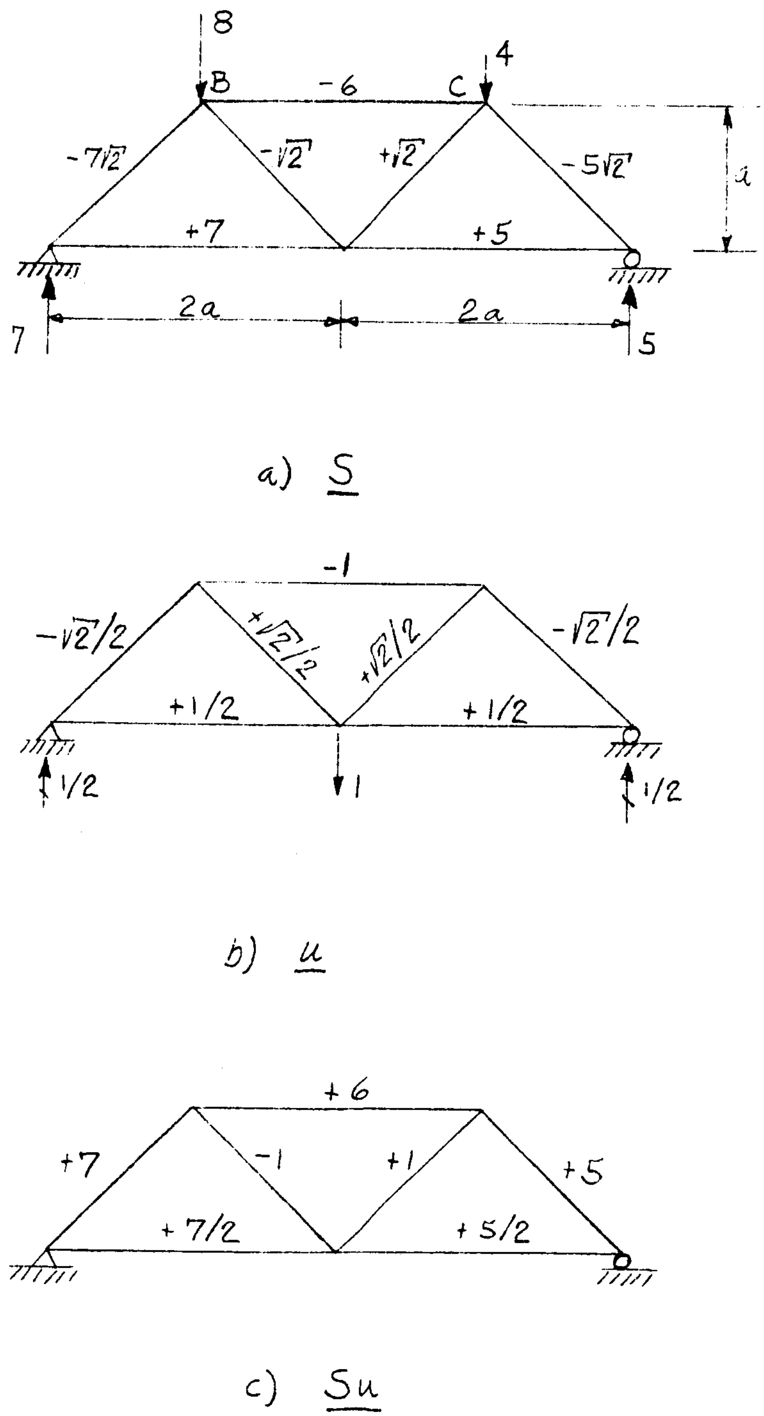


Fig. 3 SIMPLY SUPPORTED TRUSS

Some of the properties of the truss shown in Fig. 3 are given in Table 1.

TABLE 1

TRUSS PROPERTIES

Member	Designation	Specified Area	Specified Minimum Area	Su	L	$\sqrt{Su}$	$L\sqrt{Su}$	SuL/A_m
AB	1	0.5		7	$\sqrt{2}\ a$	2.646	3.742a	$14\sqrt{2}a$
AG	2		0.5	7/2	2a	1.871	3.742a	14a
BC	3		1.0	6	2a	2.449	4.899a	12a
BG	4		0.2	-1	$\sqrt{2}\ a$	-----	------	$-5\sqrt{2}a$
CD	5		0.5	5	$\sqrt{2}\ a$	2.236	3.162a	$10\sqrt{2}a$
CG	6		0.2	1	$\sqrt{2}\ a$	1.000	1.414a	$5\sqrt{2}a$
DG	7		0.5	5/2	2a	1.581	3.162a	10a

The truss deflection, when specified and minimum bar areas are used, is given by the sum of the righthand column; thus,

$$\text{Deflection} = \frac{69.936\ a}{E}$$

However, the specified deflection is

$$\Delta = \frac{55.000\ a}{E}$$

<u>Compute Eq. (16)</u>:

$$\Delta - \frac{S_1u_1L_1}{E\ A_1} + \left|\frac{S_4u_4L_4}{E\ A_4}\right| = \frac{55a}{E} - \frac{14\sqrt{2}\ a}{E} + \left|\frac{-5\sqrt{2}\ a}{E}\right| > 0\ ,$$

therefore, solutions exist.

Step 1:

Set $A_4 = 0.2$ and transfer it to group c. Then

$$\sum_c \frac{S_c u_c L_c}{E\ A_c} = \frac{14\sqrt{2}\ a}{E} - \frac{5\sqrt{2}\ a}{E} = \frac{12.728\ a}{E}$$

Step 2:

$$A_o^* = \frac{\sqrt{S_o u_o}\ \sum_o (S_o u_o)^{1/2} L_o}{E\Delta - \sum_c \frac{S_c u_c L_c}{A_c}}$$

$$A_o^* = \frac{\sqrt{S_o u_o}\ (16.379)}{55.000 - 12.728} = 0.3875\ (S_o u_o)^{1/2}$$

Member	Minimum Area	A_o^*
1	0.5	------
2	0.5	0.7250
3	1.0	0.9490 ← $A_3^* < A_{3m}$
4	0.2	------
5	0.5	0.86645
6	0.2	0.3875
7	0.5	0.6126

Step 3:

Set $A_3 = 1.0$ and transfer to group c. Then

$$\sum_c \frac{S_c u_c L_c}{E_c A_c} = \frac{12.728\ a}{E} + \frac{12.000\ a}{E} = \frac{24.728\ a}{E}$$

Step 4:

Return to Step 2.

$$A_o^* = \frac{\sqrt{S_o u_o}(11.480)}{55.000 - 24.728} = 0.37923\ (S_o u_o)^{1/2}$$

Member	Minimum Area	A_o^*
1	0.5	-------
2	0.5	0.70954
3	1.0	-------
4	0.2	-------
5	0.5	0.84796
6	0.2	0.37923
7	0.5	0.59956

Note that $A_o^* > A_m$ and the procedure ends.

C. Summary

(1) Member	(2) Area (A_m or A_o^*)	(3) SuL/A	(4) AL	(5) A_mL
1	0.5000	19.79900a	0.7071a	0.7071a
2	0.7095	9.86554a	1.4190a	1.0000a
3	1.0000	12.00000a	2.0000a	2.0000a
4	0.2000	-7.07105a	0.2828a	0.2828a
5	0.8480	8.33889a	1.1991a	0.7071a
6	0.3792	3.72916a	0.5362a	0.2828a
7	0.5996	8.33944a	1.1992a	1.0000a
		55.00098a	7.3434a	5.9798a

Column (2) lists the optimum areas. Using these areas the deflection is computed in column (3) as a check on the computations. The volume of material in the optimum truss is given in column (4), and the volume of material based on minimum areas is given in column (5).

Step 3 of the design procedure is the only step which requires elaboration. When an optimum area A_j^* is increased to its minimum value, the optimum values of all the other member areas are affected. If such an increase can cause some of the optimum values of the open members to increase, our design procedure breaks down. There would always be the possibility that we had assigned a minimum value of area to a member which might have required a larger area after other member areas had been increased to their minimum values in accordance with Step 3. We must, therefore, show that an increase in any optimum area value will decrease all of the other optimum area values.

Assume that Steps 1 and 2 have been performed. Now consider any open members, say the i th. If we fix that area of this member at A_i, the expression for the remaining optimum areas is found by appropriately modifying Eq. (9); thus,

$$A_o^* = \frac{\sqrt{\dfrac{S_o u_o}{\rho_o E_o}} \sum\limits_{o-i} \left(\dfrac{S_o u_o \rho_o}{E_o}\right)^{1/2} L_o}{\Delta - \sum\limits_{c} \dfrac{S_c u_c L_c}{A_c E_c} - \dfrac{S_i u_i L_i}{A_i E_i}} \tag{17}$$

where the symbol $\sum\limits_{o-i}$ denotes the summation over all the open members except the i th. When the area A_i is fixed at its optimum value A_i^*, Eq. (17) reduces to Eq. (9). When the area A_i is fixed at a value greater than A_i^*, we shall denote the values given by Eq. (17) as A_o^{**}. We must show that $A_o^{**} \leq A_o^*$ or that

$$\frac{\sqrt{\dfrac{S_o u_o}{\rho_o E_o}} \sum\limits_{o-i} \left(\dfrac{S_o u_o \rho_o}{E_o}\right)^{1/2} L_o}{\Delta - \sum\limits_{c} \dfrac{S_c u_c L_c}{A_c E_c} - \dfrac{S_i u_i L_i}{A_i E_i}} \leq \frac{\sqrt{\dfrac{S_o u_o}{\rho_o E_o}} \sum\limits_{o-i} \left(\dfrac{S_o u_o \rho_o}{E_o}\right)^{1/2} L_o}{\Delta - \sum\limits_{c} \dfrac{S_c u_c L_c}{A_c E_c} - \dfrac{S_i u_i L_i}{A_i^* E_i}} \tag{18}$$

Since $A_i \geq A_i^*$ the denominator on the left is greater than or equal to the denominator on the right and the inequality obviously holds. Hence Step 3 is justified.

IV. OPTIMUM TRUSS CONFIGURATIONS

The equivalence of maximum rigidity with minimum total strain energy or uniform stress for a given volume of material has been suggested by a number of authors (Ref. 2,3,4,5,6) beginning with H. R. Cox in 1936 and continuing with Richards and Chan in 1966. Saelman (Ref.7) demonstrated in 1958 that these conditions do not result in maximum stiffness for the torsion problem; Barnett (Ref. 1,8) proved a similar result for beams and established the circumstances under which minimum deflection designs display uniform maximum fiber stresses. We shall begin this section by re-emphasizing the previously established relationship between optimum stiffness/weight trusses and uniform stress trusses.

For statically determinate trusses proportioned entirely on the basis of stiffness, the minimum weight W^* (Case 4) is given by Eq. (10). The corresponding weight of the uniform stress truss W_c is given by Eq. (31) in Appendix B. When all bar areas are open and only one material is used, we wish to show that $W^* \leq W_c$. Thus,

$$W^* - W_c = \left[\sum_o \sqrt{\frac{S_o u_o L_o}{\Delta |S_o|}} \sqrt{\frac{L_o |S_o|}{E/\rho}}\right]^2 - \sum_o \frac{S_o u_o L_o}{\Delta |S_o|} \sum_o \frac{L_o |S_o|}{E/\rho} \leq 0 \tag{19}$$

From Schwarz's inequality we conclude that $W^* \leq W_c$ and that the equality holds if and only if u_o is proportional to S_o, i.e., $bu_o + cS_o = 0$ where b and c are constants.

We shall first consider the case where the actual and virtual loadings are proportional. Here, S_o is proportional to u_o and the minimum deflection truss is uniformly stressed.

Its weight can be found from either Eq. (10) or Eq. (31) when we take $S_o/u_o = k$ and only one material is used.

$$W^* = W_c = \frac{(k/\Delta)}{(E/\rho)} \left[\sum_o |S_o| L_o \right]^2 \tag{20}$$

This equation represents the lowest possible weight for a statically determinate truss of given configuration which is designed for a specified deflection Δ (or stiffness k/Δ). If we wish to select the optimum truss configurations from all possible minimum weight candidates, we must choose those which minimize the quantity shown in the brackets of Eq. (20).

In 1904, Michell (Ref.9) developed the conditions for minimizing the quantity $\sum_o |S_o| L_o$. The associated minimum weight structures are usually found to be statically determinate; however, hyperstatic Michell structures may sometimes occur. If this should happen, it is always possible to find an equal weight statically determinate Michell structure. Referring to the literature written in English, this is guaranteed by the theorems of Sved (Ref. 10) and Barta (Ref. 11) which state that in pin-jointed plane or space structures of n bars involving r redundancies, it is possible to obtain a statically determinate structure which yields the least possible weight by removing r properly chosen redundant bars from the given network. The theorem holds for fixed, not necessarily equal, permissible stresses in tension and compression.

To summarize, when the actual and virtual loadings on a truss are proportional, the optimum stiffness/weight truss is given by a Michell structure, either statically determinate or indeterminate, designed for equal magnitude tensile and compressive stresses. Such a structure minimizes the total strain energy as shown by Richards and Chan (Ref.5) who propose this condition arbitrarily as a general stiffness criterion.

The authors H. L. Cox (Ref. 3) and Hemp (Ref. 4) also adopt this minimum strain energy argument; but, they incorrectly propose the general Michell structure without requiring that all stresses have the same magnitude.

Very few situations arise where the actual and virtual loadings are proportional. Such cases are encountered when a single concentrated load acts on a truss and the deflection under the load is minimized. Examples of this case are furnished by the Michell structures shown in Figs 4a and 4b which minimize respectively the central and tip deflections. We note that the length to depth ratio for these optimum members may be impractically large; for the optimum simply supported beam $L/d = \sqrt{2}$. For such problems the work done by the single force F acting through the deflection δ must equal the strain energy U; thus,

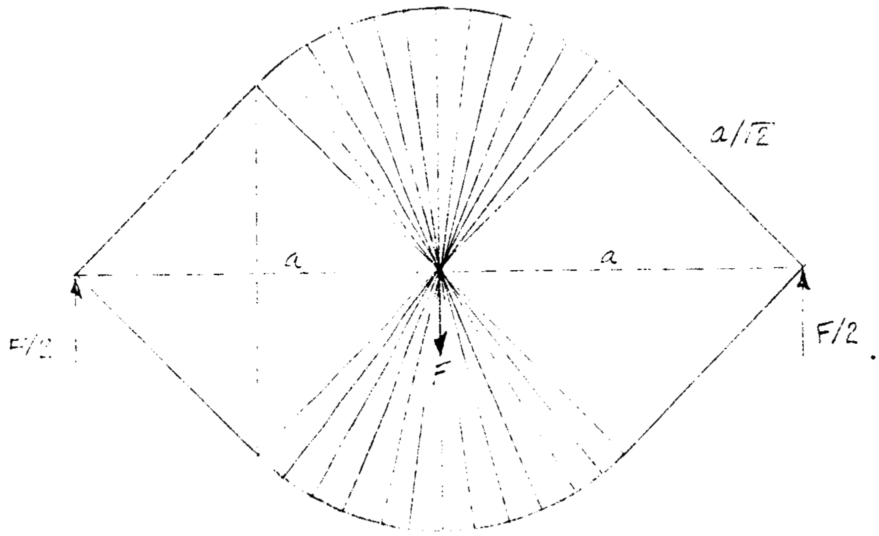
$$\frac{F\,\delta}{2} = U$$

In this simple situation it is clear that the structure minimizing U will also minimize δ.

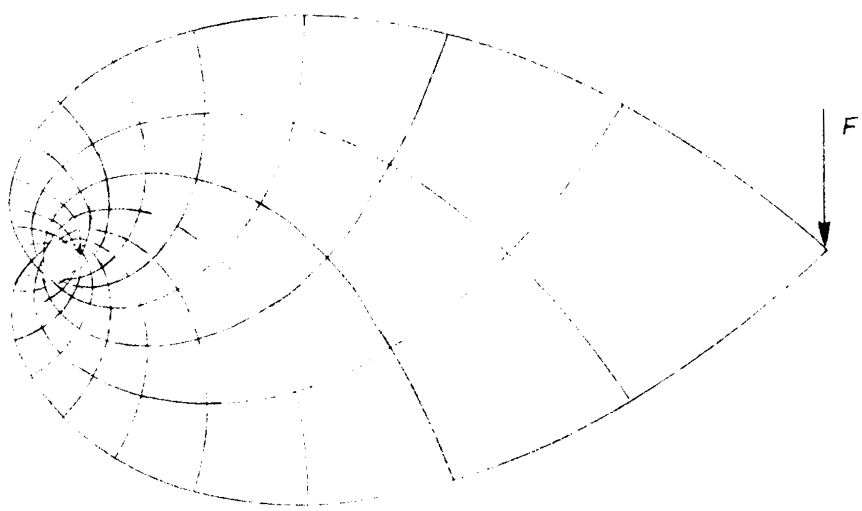
If several loads of the same magnitude F act on a truss and δ_i represents the deflection under the i th load in the direction of its action, the optimum may again lead to a Michell structure. Equating the strain energy U to the work done by the forces F we obtain

$$F \sum_i \delta_i = U$$

Consequently, a structure which minimizes the strain energy will also minimize the sum of the displacements under the forces F. To formulate this problem using virtual loads, we note that the deflection formula, Eq. (1), used in the unit load method when



a. Simply Supported Beam



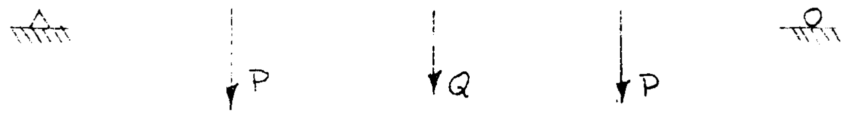
b. Cantilever Beam

Fig. 4 OPTIMUM MICHELL STRUCTURES

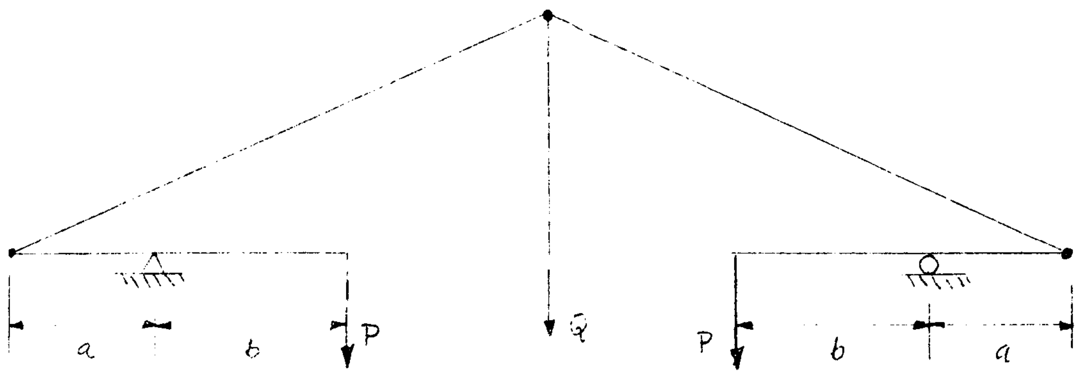
several unit loads are placed on a structure predicts the sum of the deflections which occur under the unit loads. Thus, in this case, we would place a unit load to correspond to each load F. The resulting virtual loading would be proportional to the actual loading, Δ would be interpreted as $\sum_i \delta_i$, and the optimum truss would be a uniformly stressed Michell structure. Now, if we wish to minimize the sum of the deflections under a set of loads of unequal magnitudes, we observe that the actual and virtual loadings are no longer proportional and that the optimum truss cannot be a Michell structure.

The two cases leading to Michell structures which we have just examined are probably the only situations where S_o and u_o are proportional. In all other problems minimum strain energy structures will not provide optimum stiffness/weight trusses. For such problems it does not appear to be difficult to choose bar arrangements which lead to the degenerate case described under Case 3 in Section II. Here, we recall that positive, negative, or zero deflections can be obtained with an infinite number of trusses of vanishingly small weight. The structure shown in Fig. 5a provides an example of a degenerate truss design for a single concentrated load which does not act at the node where we are interested in the deflection.

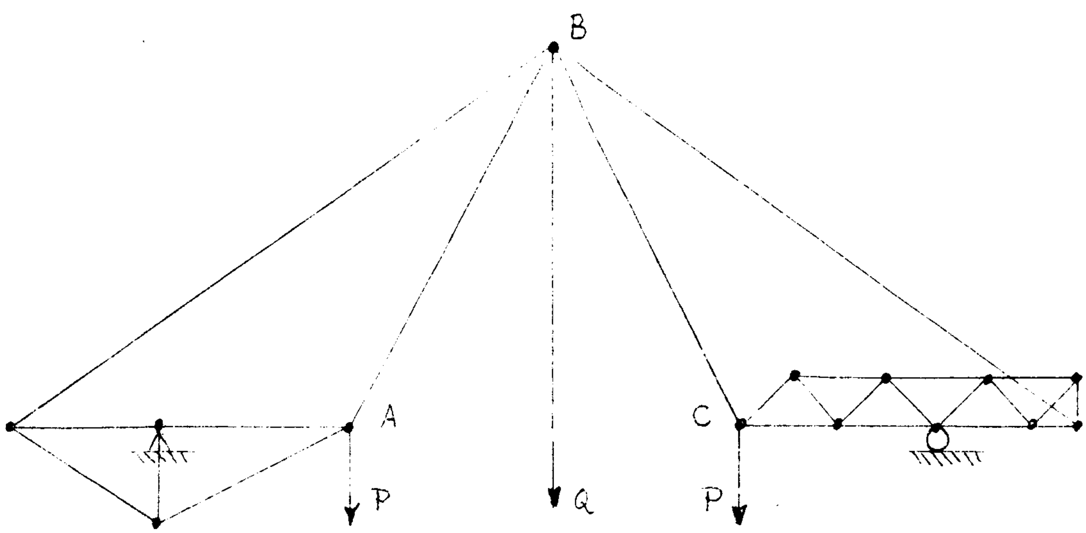
Another example is illustrated in Fig. 5b where it is required that we minimize the central deflection in a truss under three symmetrically located forces. Examining Fig. 5a makes it clear that a virtual unit load placed in the center of the span cannot be proportional to the three forces shown; consequently, the optimum truss will not be a Michell structure. Observe that the linkage shown in Fig. 5b will provide an upward force and movement at the center node to counteract the load Q. Adding bars to this mechanism, we can obtain the statically determinate truss shown in Fig. 5c.



a. Specified Loading



b. Linkage



c. Degenerate Truss

Fig. 5 OPTIMUM TRUSS DESIGN (Virtual and Actual Loadings not Proportional)

A simple static analysis indicates that the members AB and BC in this truss will provide negative values of the product Su when the dimension a is adjusted so that

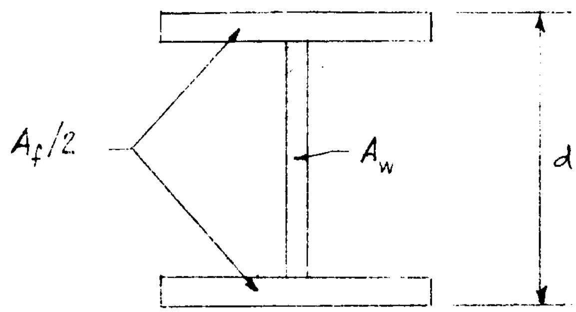
$$Q < \frac{2b}{a} P \qquad a \neq 0$$

Hence, the conditions of Case 3 are realized; namely, both positive and negative values of $S_o u_o$ occur in the same truss. The bar areas may be adjusted in an infinite number of ways to obtain any central deflection desired. We note in closing that when the actual and virtual loadings are proportional, a negative deflection would violate the conservation of energy.

APPENDIX A*

OPTIMUM STIFFNESS - WEIGHT BEAMS

A detailed development of minimum weight beams designed for deflection may be found in Ref. 1. The relatively exact treatment given there for I-beams, unfortunately, masks the influence of some of the important beam parameters on optimum beam weight. Here, we obtain a very simple formula for minimum beam weight by using approximate expressions for the moment of inertia and shape factor.



I-Beam Section

The moment of inertia and shape factor for the section shown in the above figure are,

$$I \doteq \frac{d^2}{12}\,(3A_f + A_w)$$

$$\alpha/A \doteq 1/A_w \; \ldots \text{ (See Ref. 13).}$$

The deflection of any statically determinate beam is given by

$$\Delta = \int_S \left(\frac{Mm}{EI} + \frac{\alpha Vv}{GA}\right) dx = \int_S \left[\frac{12Mm}{Ed^2(3A_f + A_w)} + \frac{Vv}{GA_w}\right] dx \qquad (21)$$

*The material in this appendix was taken from the paper by Barnett (Ref.12) and is included here for the sake of completeness.

where M is the bending moment, V is the shear, m is the virtual moment and v is the virtual shear caused by a unit load placed where the deflection is desired, G is the shear modulus and the integral is taken over the span S. The weight of the beam is given by,

$$W = \int_S \rho(A_f + A_w)\, dx \tag{22}$$

Using variational calculus, the conditions for minimizing the weight W subject to the requirement that Δ equal is specified constant are

$$\frac{\partial}{\partial A_f}\left\{\rho(A_f + A_w) + \gamma\left[\frac{12Mm}{Ed^2(3A_f + A_w)} + \frac{Vv}{GA_w}\right]\right\} = 0 \tag{23}$$

$$\frac{\partial}{\partial A_w}\left\{\rho(A_f + A_w) + \gamma\left[\frac{12Mm}{Ed^2(3A_f + A_w)} + \frac{Vv}{GA_w}\right]\right\} = 0 \tag{24}$$

where γ is a constand multiplier. Performing the operations in Eq. (23) and (24), and eliminating γ with Eq. (21), the optimum area distributions become

$$A_w = \sqrt{\frac{3Vv}{2\rho G}}\,\frac{1}{\Delta}\int_S\left(\sqrt{\frac{4Mm\rho}{Ed^2}} + \sqrt{\frac{2Vv\rho}{3G}}\right)dx \tag{25}$$

$$A_f = \sqrt{\frac{4Mm}{\rho Ed^2}} - \sqrt{\frac{Vv}{6\rho G}}\;\frac{1}{\Delta}\int_S\left(\sqrt{\frac{4Mm\rho}{Ed^2}} + \sqrt{\frac{2Vv\rho}{3G}}\right)dx \tag{26}$$

Substituting these areas into Eq. (22), we obtain an expression for the minimum weight beam.

$$W_b = \frac{1}{\Delta(E/\rho)}\left[\int_S\left(\sqrt{\frac{4Mm}{d^2}} + \sqrt{\frac{2Vv\beta^2}{3}}\right)dx\right]^2 \tag{27}$$

where

$$\beta^2 \equiv E/G \tag{28}$$

APPENDIX B

CONSTANT STRESS TRUSSES

The weight of a uniform stress truss designed for deflection often approximates and sometimes equals the weight of a minimum weight truss. The bar areas of such a truss are given by

$$A_o = \frac{|S_o|}{\sigma} \tag{29}$$

where σ is a constant. By substituting A_o into Eq. (1), the value of σ may be found for any specified deflection; thus

$$\sigma = \frac{\Delta}{\sum_o \frac{S_o}{|S_o|} \frac{u_o L_o}{E_o}} \tag{30}$$

Using the expressions for A_o and σ, the weight of a constant stress truss W_c becomes,

$$W_c = \frac{1}{\Delta} \left(\sum_o \rho_o L_o |S_o| \right) \left(\sum_o \frac{S_o}{|S_o|} \frac{u_o L_o}{E_o} \right) \tag{31}$$

APPENDIX C

1. End Loaded Cantilevers

In this appendix, we shall compute the weight of a minimum weight, constant depth, cantilever truss and beam subjected to a concentrated end load. The truss geometry is defined in Fig. 6 .

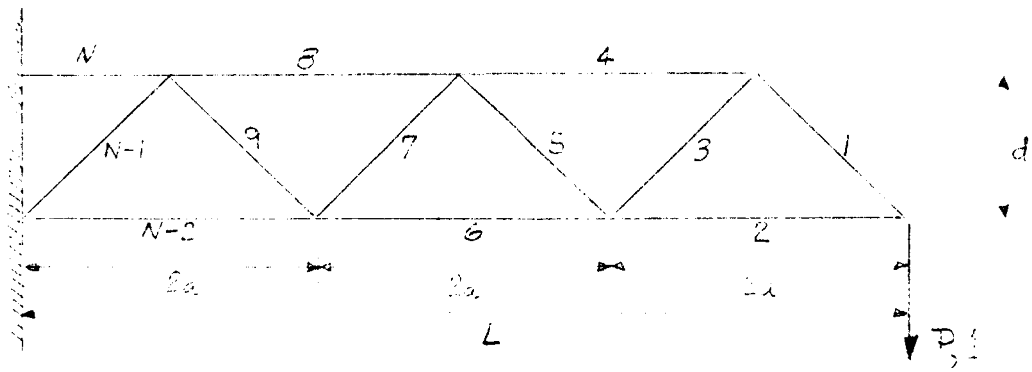


Fig. 6 END LOADED CANTILEVER TRUSS

Expressions are given below for the actual and virtual bar forces and the bar lengths.

$$S_n = (-1)^{\frac{n+3}{2}} P \sqrt{1 + \alpha^2}$$

$$u_n = (-1)^{\frac{n+3}{2}} \sqrt{1 + \alpha^2} \qquad n = 1,3,5,\ldots N-1$$

$$S_n = (-1)^{\frac{n}{2}} \frac{P\alpha n}{2}$$

$$u_n = (-1)^{\frac{n}{2}} \frac{\alpha n}{2} \qquad n = 2,4,6,\ldots N$$

$$L_n = d \sqrt{1 + \alpha^2} \qquad n = 1,3,5,\ldots N-1$$

$$L_n = 2a \qquad n = 2,4,6,\ldots N-2$$

$$L_n = a \qquad n = N$$

where

$$\alpha \equiv a/d$$

$$N = 2(L/a)$$

It is noted that the ratio $S/u = P$ for each of the truss members. Since this ratio is constant throughout the truss, the minimum weight design and the uniform stress design are equivalent.

Using the expressions for S_n, u_n, L_n, the truss weight may be computed from Eq. (10).

$$W^* = \frac{1}{\Delta}\left[\sum_o \left(\frac{S_o u_o}{E_o/\rho_o}\right)^{1/2} L_o\right]^2$$

$$W^* = \frac{1}{\Delta}\left[\sum_{n=1,3}^{N-1} \sqrt{\frac{P}{E/\rho}}\,(1+\alpha^2)d + \sum_{n=2,4}^{N-2} \sqrt{\frac{P}{E/\rho}}\,\frac{n\alpha}{2}\,2a + \sqrt{\frac{P}{E/\rho}}\,\frac{N\alpha}{2}\,a\right]^2$$

$$W^* = \frac{PL^2}{\Delta(E/\rho)}\left(\frac{1+\alpha^2}{\alpha} + L/d\right)^2$$

The minimum of this expression occurs when the web members are placed at 45 degree angles, i.e., $\alpha = 1$. The variation of W^* for angles near 45 degrees is quite small as can be seen from Fig. 7. For $\alpha = 1$, W^* becomes

$$W^* = \frac{PL^2}{\Delta(E/\rho)}(2 + L/d)^2 \tag{32}$$

If an I-beam is used instead of a truss, its weight is computed from Eq. (27) when

$$M = Px$$

$$m = x$$

$$V = P$$

$$v = 1$$

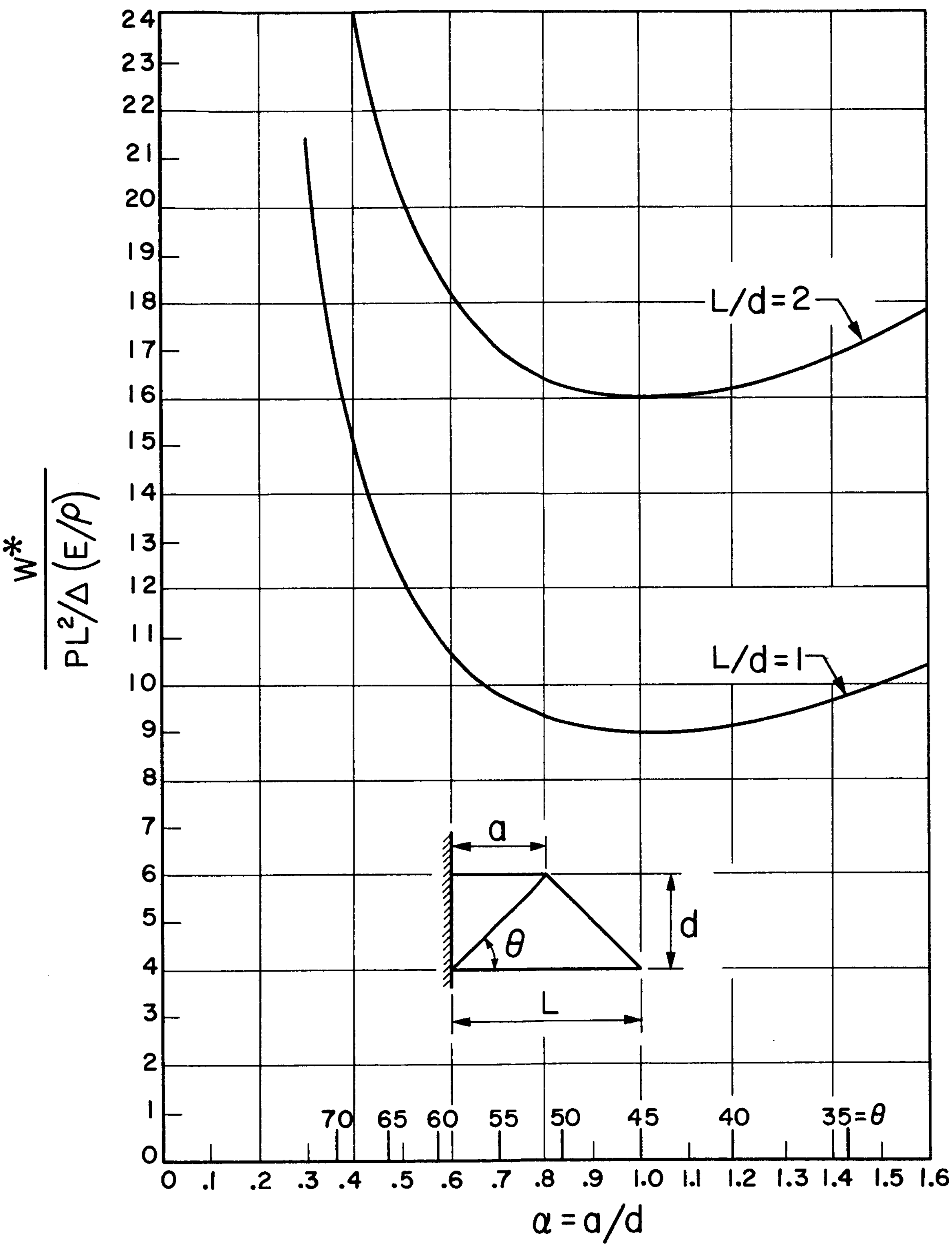


Fig. 7 VARIATIONS OF TRUSS WEIGHT WITH WEB ANGLE

and x is measured from the free end. Thus,

$$W_b = \frac{1}{\Delta(E/\rho)} \left[\int_o^L \sqrt{\frac{4Px^2}{d^2}} + \sqrt{\frac{2P\beta^2}{3}} \; dx \right]^2$$

$$W_b = \frac{1}{\Delta(E/\rho)} \left[(L/d)^2 + \sqrt{\frac{8}{3}}\, \beta(L/d) + \frac{2}{3}\, \beta^2 \right] \qquad (33)$$

The ratio of beam to truss weight is found from Eq. (32) and (33).

$$\frac{W_{beam}}{W_{truss}} = \frac{(L/d)^2 + \sqrt{\frac{8}{3}}\, \beta(L/d) + \frac{2}{3}\, \beta^2}{(2 + L/d)^2} = \frac{(L/d)^2 + 2.580\ (L/d) + 1.667}{(2 + L/d)^2} \qquad (34)$$

for $\beta^2 = 2.5$. This ratio is plotted against L/d in Fig. 2.

2. Cantilever Under a Triangular Loading

The truss defined in Fig. 8 shall be designed for minimum weight and uniform stress. The resulting deflection designs will be compared to a minimum weight, constant depth, I-beam of equivalent stiffness.

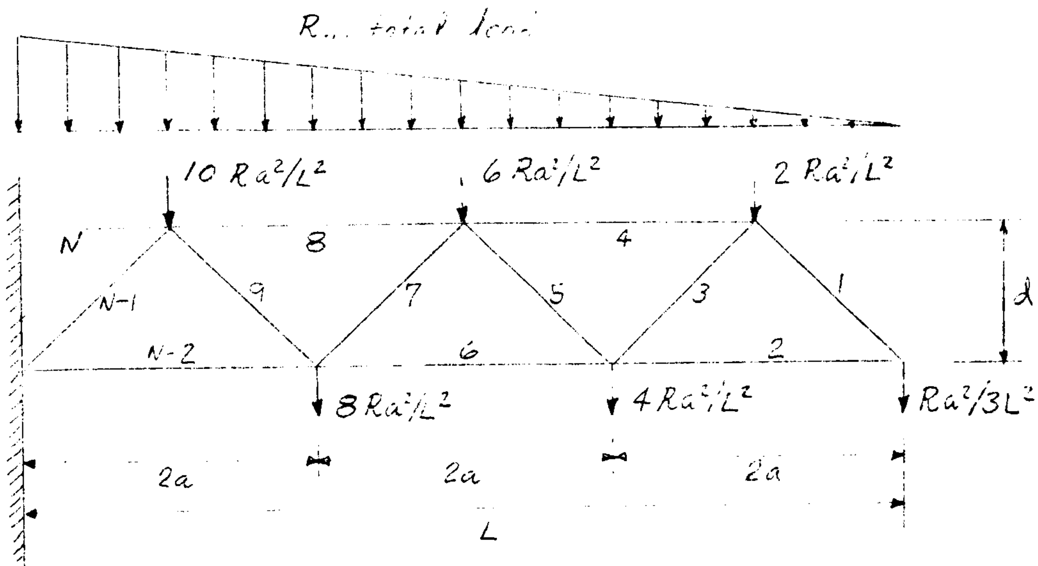


Fig. 8 TRUSS UNDER A TRIANGULAR LOADING

The actual and virtual bar stresses and the member lengths for the truss are given by the following expressions:

$$S_n = (-1)^{\frac{n+3}{2}} \frac{Ra^2}{L^2} \sqrt{1 + \alpha^2} \left(\frac{n^2}{4} + \frac{1}{12}\right)$$

$$u_n = (-1)^{\frac{n+3}{2}} \sqrt{1 + \alpha^2}$$

$n = 1,3,5,\ldots N-1$

$$S_n = (-1)^{n/2} \frac{Ra^2\alpha}{3L^2} \left(\frac{n}{2}\right)^3$$

$$u_n = (-1)^{n/2} \alpha\left(\frac{n}{2}\right)$$

$n = 2,4,6,\ldots N$

$$L_n = d\sqrt{1+\alpha^2} \qquad n = 1,3,5,\ldots N-1$$

$$L_n = 2a \qquad n = 2,4,6,\ldots N-2$$

$$L_n = a \qquad n = N$$

where

$$\alpha \equiv a/d$$

$$N = 2(L/a)$$

Substituting these into Eq. (10), we obtain the weight of a minimum weight truss.

$$W^* = \frac{1}{\Delta}\left[\sqrt{\frac{R}{E/\rho}}\,\frac{ad}{L}\,(1+\alpha^2)\sum_{n=1,3}^{N-1}\sqrt{\left(\frac{n}{2}\right)^2+\frac{1}{12}} + \sqrt{\frac{R}{E/\rho}}\,\frac{2\,a^2\alpha}{\sqrt{3}\,d}\sum_{n=2,4}^{N-2}\left(\frac{n}{2}\right)^2 + \sqrt{\frac{R}{E/\rho}}\,\frac{a^2\,\alpha}{\sqrt{3}\,d}\left(\frac{N}{2}\right)^2\right]^2$$

Introducing the approximation

$$\sum_{n=1,3}^{N-1}\sqrt{\left(\frac{n}{2}\right)^2+\frac{1}{12}} \doteq \sum_{n=1,3}^{N-1}\left(\frac{n}{2}\right) = \frac{L^2}{2a^2}\,,$$

W^* becomes

$$W^* = \frac{RL^2}{27\Delta(E/\rho)}\left[\frac{\alpha^2}{(L/d)} + \frac{3\sqrt{3}}{2\alpha}\,(1+\alpha^2) + 2(L/d)\right]^2 \qquad (35)$$

For this loading, it is clear that the optimum web angle is not 45 degrees. For $L/d = 1$, $\alpha \approx 0.79$; as $L/d \to \infty$, $\alpha \to 1$. The weight of a uniform or constant stress truss is found from Eq. (31). Using the expressions for S_n, u_n, and L_n, W_c becomes

$$W_c = \frac{1}{\Delta(E/\rho)} \left[\sum_{n=1,3}^{N-1} \frac{Ra^2 d}{12L^2} (1 + \alpha^2)(3n^2 + 1) + \sum_{n=2,4}^{N-2} \frac{Ra^2\alpha}{3L^2} \left(\frac{n}{2}\right)^2 2a + \frac{Ra^2\alpha}{3L^2} \left(\frac{N}{2}\right)^2 a \right] \cdot \left[\sum_{n=1,3}^{N-1} d(1 + \alpha^2) + \sum_{n=2,4}^{N-2} \frac{n\alpha}{2} 2a + \frac{N\alpha}{2} a \right] \tag{36}$$

$$W_c = \frac{RL^2}{6\Delta(E/\rho)} \left[\frac{\alpha^3}{L/d} + 2\alpha^2 + \frac{\alpha}{L/d} + 2\alpha(L/d) + (L/d)^2 + 3 + \frac{3}{\alpha}(L/d) + \frac{2}{\alpha^2} \right]$$

The ratio of the weights of the minimum weight truss and the constant stress truss is found from Eq. (35) and (36). For $\alpha = 1$ it becomes

$$\frac{W_{min.}}{W_{const.}} = \frac{\frac{1}{27}\left[3\sqrt{3} + 2(L/d) + \frac{1}{(L/d)}\right]^2}{\frac{1}{6}\left[(L/d)^2 + 5(L/d) + 7 + \frac{2}{(L/d)}\right]} \tag{37}$$

This ratio is plotted against L/d in Fig. 2.

The following expressions give the actual and virtual bending moments and shear acting on an I-beam subjected to a triangular loading.

$$M = \frac{R\ x^3}{3L^2}$$

$$m = x$$

$$V = \frac{R\ x^2}{L^2}$$

$$v = 1$$

where x is measured from the free end of the cantilever. Using these relationships, the weight of a minimum weight I-beam can be computed from Eq. (27); thus,

$$W_b = \frac{1}{\Delta(E/\rho)} \left[\int_o^L \left(\sqrt{\frac{4\ Rx^4}{d^2\, 3L^2}} + \sqrt{\frac{2Rx^2\beta^2}{3L^2}} \right) dx \right]$$

$$W_b = \frac{4RL^2}{27\Delta(E/\rho)} \left(\frac{3\sqrt{2}}{4}\, \beta + \frac{L}{d} \right)^2 \tag{38}$$

The ratio of the weights of the minimum weight beam and the minimum weight truss can be found from Eq. (35) and (38). For $\alpha = 1$ and $\beta^2 = 2.5$, the ratio becomes

$$\frac{W_{beam}}{W_{truss}} = \frac{4(1.667 + L/d)^2}{\left[5.196 + 2(L/d) + \frac{1}{(L/d)} \right]^2} \tag{39}$$

This ratio is plotted against L/d in Fig. 2.

APPENDIX D

NOTATION

A	Bar area, or beam area
A^*	Optimum bar area
A_f	Flange areas of an I-beam
A_m	Minimum allowable area of a truss member
A_w	Web area of an I-beam
A^{**}	Defined where used
A_1	Defined where used
A_2	Defined where used
a	Half the distance between panel points of a truss, or overhang distance
b	Constant, or linear dimension
d	Truss or beam depth
E	Modulus of elasticity
F	$\equiv W - W^*$
G	Modulus of rigidity
I	Moment of inertia
k	Constant
L	Length of a truss member, or span of a beam or truss
M	Bending moment
m	Virtual bending moment
N	Total number of truss members
n	The number of a truss member; integral variable
P	Concentrated end load
R	Total triangular load

S Beam span; Bar force

u Virtual bar force

V External shear force

v Virtual shear force

W Beam or truss weight

W^* Minimum truss weight

W_b Minimum beam weight

W_c Weight of a constant stress truss

W_t Minimum truss weight

x Independent variable

$\alpha \equiv a/d$...also, shape factor

γ Constant multiplier

ρ Weight density

σ Stress level

θ Web angle

Δ Specified deflection or deflection

λ Lagrangian multiplier

c Subscript denoting closed or defined members

o Subscript denoting open members

i Subscript denoting a specific truss member

$\beta \equiv \sqrt{E/G}$

APPENDIX E

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